Parity biases in partitions and restricted partitions

Koustav Banerjee[1], Sreerupa Bhattacharjee[2], Manosij Ghosh Dastidar[3], Pankaj Jyoti Mahanta[4], and Manjil P. Saikia[5]

[1]Johannes Kepler University, koustav.banerjee@risc.uni-linz.ac.at, [2]University of Lethbridge, bhatttacharjee.sreerupa@gmail.com, [3]Technische Universität Wien, gdmanosij@gmail.com, [4]Gonit Sora, pankaj@gonitsora.com, [5]Cardiff University manjil@saikia.in

Parity bias

The tendency of partitions to have more parts of a particular parity than the other is often called *parity bias*.

Example: Let $q_o(n)$ (resp. $q_e(n)$) denote the number of partitions of n with more odd parts (resp. even parts) than even parts (resp. odd parts) where the smallest part is at least 2. Following are the partitions of 8 where the smallest part is at least 2:

$$8, 6+2, 5+3, 4+4, 4+2+2,$$
$$3+3+2, 2+2+2+2.$$

So, $q_o(8) = 2$, and $q_e(8) = 5$. That is $q_o(8) < q_e(8)$. In fact, $q_o(n) < q_e(n)$ for all $n > 7$.

Definitions

A partition λ of a non-negative integer n is an integer sequence $(\lambda_1, \ldots, \lambda_\ell)$ such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0$ and $\sum_{i=1}^{\ell} \lambda_i = n$. We say that λ is a partition of n, denoted by $\lambda \vdash n$ and $\sum_{i=1}^{\ell} \lambda_i = n$. The set of partition of n is denoted by $P(n)$ and $|P(n)| = p(n)$. For $\lambda \vdash n$, we define $a(\lambda)$ to be the largest part of λ, $\ell(\lambda)$ to be the total number of parts of λ and $\mathrm{mult}_\lambda(\lambda_i) := m_i$ to be the multiplicity of the part λ_i in λ. We also use $\lambda = (\lambda_1^{m_1} \ldots \lambda_\ell^{m_\ell})$ as an alternative notation for partition. For $\lambda \vdash n$ with $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ and $\mu \vdash m$ with $\mu = (\mu_1, \ldots, \mu_{\ell'})$, define the union $\lambda \cup \mu \vdash m + n$ to be the partition with parts $\{\lambda_i, \mu_j\}$ arranged in non-increasing order. For a partition $\lambda \vdash n$, we split λ into λ_e and λ_o respectively into even and odd parts; i.e., $\lambda = \lambda_e \cup \lambda_o$. We denote by $\ell_e(\lambda)$ (resp. $\ell_o(\lambda)$) to be the number of even parts (resp. odd parts) of λ and $\ell(\lambda) = \ell_e(\lambda) + \ell_o(\lambda)$.

More definitions

$$
\begin{aligned}
D(n) &:= \{\lambda \in P(n) : \mathrm{mult}_\lambda(\lambda_i) = 1 \text{ for all } i\},\\
P_e(n) &:= \{\lambda \in P(n) : \ell_e(\lambda) > \ell_o(\lambda)\},\\
P_o(n) &:= \{\lambda \in P(n) : \ell_o(\lambda) > \ell_e(\lambda)\},\\
D_e(n) &:= P_e(n) \cap D(n),\\
D_o(n) &:= P_o(n) \cap D(n),\\
Q(n) &:= \{\lambda \in P(n) : \lambda_i \neq 1 \text{ for all } i\},\\
Q_e(n) &:= \{\lambda \in Q(n) : \ell_e(\lambda) > \ell_o(\lambda)\},\\
Q_o(n) &:= \{\lambda \in Q(n) : \ell_o(\lambda) > \ell_e(\lambda)\},\\
DQ_e(n) &:= Q_e(n) \cap D(n),\\
\text{and}\quad DQ_o(n) &:= Q_o(n) \cap D(n).
\end{aligned}
$$

For a nonempty set $S \subsetneq \mathbb{Z}_{\geq 0}$,

$$
\begin{aligned}
P_e^S(n) &:= \{\lambda \in P_e(n) : \lambda_i \notin S\}\\
\text{and}\quad P_o^S(n) &:= \{\lambda \in P_o(n) : \lambda_i \notin S\}.
\end{aligned}
$$

For all the sets defined above, their cardinalities will be denoted by the lower case letters. For instance, $|P_e(n)| = p_e(n)$, $|DQ_e(n)| = dq_e(n)$ and so on.

Theorems

We prove the following theorems combinatorially:

Theorem 1 (Theorem 1, [2]). *For all positive integers $n \neq 2$, we have $p_o(n) > p_e(n)$.*

Theorem 2 (Conjectured, [2]). *For all positive integers $n > 19$, we have $d_o(n) > d_e(n)$.*

Theorem 3. *For all positive integers $n > 7$, we have $q_o(n) < q_e(n)$.*

Theorem 4. *For all $n \geq 1$ we have $p_o^{\{2\}}(n) > p_e^{\{2\}}(n)$.*

Theorem 5. *If $S = \{1, 2\}$, then for all integers $n > 8$, we have $p_o^S(n) > p_e^S(n)$.*

2020 Mathematics Subject Classification. 05A17, 05A20, 11P83.
Key words and phrases. Combinatorial inequalities, Partitions, Parity of parts, Restricted Partitions.

The fundamental principle behind proofs of Theorems

To prove the Theorem 1, we consider

$$
\begin{aligned}
G_e^0(n) &:= \{\lambda \in P_e(n) : \ell_e(\lambda) - \ell_o(\lambda) = 1 \text{ and } a(\lambda) \equiv 0 \pmod 2\},\\
\overline{G_e^0}(n) &:= \{\lambda \in G_e^0(n) : \lambda_3 \geq 3\},\\
G_e^1(n) &:= \{\lambda \in P_e(n) : \ell_e(\lambda) - \ell_o(\lambda) = 1 \text{ and } a(\lambda) \equiv 1 \pmod 2\},\\
G_e^2(n) &:= \{\lambda \in P_e(n) : \ell_e(\lambda) - \ell_o(\lambda) \geq 2\},\\
\text{and}\quad G_e(n) &:= G_e^1(n) \cup G_e^2(n).
\end{aligned}
$$

We split the set $G_e(n)$ into the parity of length of partition as $G_e(n) = G_{e,0}(n) \cup G_{e,1}(n)$ with $G_{e,0}(n) = \{\lambda \in G_e(n) : \ell(\lambda) \equiv 0 \pmod 2\}$, $G_{e,1}(n) = \{\lambda \in G_e(n) : \ell(\lambda) \equiv 1 \pmod 2\}$ and let $\overline{G_e}(n) := G_{e,0}(n) \cup G_{e,1}(n) \cup \overline{G_e^0}(n)$. Therefore,

$$P_e(n) \setminus \overline{G_e}(n) = \{\lambda \in G_e^0(n) : 0 \leq \lambda_3 \leq 2\}.$$

We construct a map $f : \overline{G_e}(n) \to P_o(n)$ by defining maps $f|_{G_{e,0}(n)} = f_1$, $f|_{G_{e,1}(n)} = f_2$ and $f|_{\overline{G_e^0}(n)} = f_3$ such that $\{f_i\}_{1\leq i\leq 3}$ are injective with the following properties

- $f_1(G_{e,0}(n)) \cap f_2(G_{e,1}(n)) = \emptyset$,
- $f_1(G_{e,0}(n)) \cap f_3(\overline{G_e^0}(n)) = \emptyset$, and
- $f_2(G_{e,1}(n)) \cap f_3(\overline{G_e^0}(n)) = \emptyset$,

so as to conclude the map f is injective. Then we will choose a subset $\overline{P_o}(n) \subsetneq P_o(n) \setminus f(\overline{G_e}(n))$ with $|\overline{P_o}(n)| > |P_e(n) \setminus \overline{G_e}(n)|$.

Problems

Problem 1. *For all $m > 6$ we have $dq_o(2m) > dq_e(2m)$, and*

$$dq_o(2m+1) < dq_e(2m+1).$$

Problem 2. *For all $k > 2$ we have $p_o^{\{k\}}(n) > p_e^{\{k\}}(n)$ and $p_e^{\{1,k\}}(n) > p_o^{\{1,k\}}(n)$, for all $n > N(k)$, for some constant $N(k)$, depending on k. Moreover, it would be worthwhile to understand the threshold $N(k)$ asymptotically.*

References

[1] Koustav Banerjee, Sreerupa Bhattacharjee, Manosij Ghosh Dastidar, Pankaj Jyoti Mahanta, and Manjil P Saikia. Parity biases in partitions and restricted partitions. *European Journal of Combinatorics*, 103: 103522, 2022. doi: 10.1016/j.ejc.2022.103522.

[2] Byungchan Kim, Eunmi Kim, and Jeremy Lovejoy. Parity bias in partitions. *European Journal of Combinatorics*, 89:103159, 2020. doi: 10.1016/j.ejc.2020.103159.